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Group theoretical analysis of 600-cell and 120-cell 4D polytopes with quaternions

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Abstract
600-cell {3, 3, 5} and 120-cell {5, 3, 3} four-dimensional dual polytopes relevant to quasicrystallography have been studied with the quaternionic representation of the Coxeter group $W(H_4)$. The maximal subgroups $W(SU(5)) : Z_2$ and $W(H_3) \times Z_2$ of $W(H_4)$ play important roles in the analysis of cell structures of the dual polytopes. In particular, the Weyl–Coxeter group $W(SU(4))$ is used to determine the tetrahedral cells of the polytope {3, 3, 5}, and the Coxeter group $W(H_3)$ is used for the dodecahedral cells of {5, 3, 3}. Using the Lie algebraic techniques in terms of quaternions, we explicitly construct cell structures forming the vertices of the 4D polytopes.

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1. Introduction

The non-crystallographic structures in three dimensions requires the study of the non-crystallographic Coxeter groups $W(H_3)$ of order 120 and its extension $W(H_4)$ to four-dimensional Euclidean space of order 14 400, for they generate some interest in quasicrystallography [1–3]. Moreover, the non-crystallographic lattice structures in the crystallographic lattice E_8 can be well understood by embedding $W(H_4)$ in $W(E_8)$ [4, 5], the Coxeter–Weyl group of the exceptional Lie group E_8 which seems to be playing an important role in the superstring theory [6]. The Coxeter group $W(H_3)$, isomorphic to the icosahedral group with inversion $A_5 \times Z_2$, represents the symmetry of the C_{60} molecule [7], the Al–Mn alloys [8] displaying quasicrystallography and some viruses [9]. Although of the regular 4D polytopes {3, 3, 5} and {5, 3, 3} have not been used in the quasicrystallography, their projection into three-dimensional Euclidean space made with the use of $W(H_3)$ and $W(SU(4))$ may introduce further insight into understanding the quasicrystallography in three dimensions [10]. The polyhedral structures in three dimensions and the polytopes (polycora) in four dimensions have been studied with the use of the Lie algebraic technique [11] and the use of quaternions. The work of [11] can be extended to the regular and semi-regular polytopes in higher dimensions using the Coxeter–Dynkin diagrams and the Coxeter–Weyl groups. In this paper, we study the cell structures of the dual polytopes {3, 3, 5} and {5, 3, 3} using the subgroups $W(SU(4))$ and $W(H_3)$ of the Coxeter group $W(H_4)$. The group $W(SU(5)) : Z_2$, an extension of the Coxeter–Weyl group $W(SU(5))$ with the Coxeter–Dynkin diagram symmetry, is one of the five maximal subgroups of the Coxeter group $W(H_4)$ [12]. Here, (:) stands for the semi-direct product. In section 2 we construct the Coxeter–Weyl group $W(SU(5))$ using its Coxeter–Dynkin diagram represented by quaternionic simple roots. We first construct the vertices of the 5-cell, the weights of the five-dimensional irreducible representation of $SU(5)$, in terms of quaternions and show how one can obtain the 600 vertices of the 120-cell represented by quaternions. We also discuss the five-fold embeddings of $W(SU(4))$ into the group $W(SU(5))$ and study the tetrahedral structures of the 600-cell polytope having 120 vertices. In section 3, we construct the group elements of $W(SU(4))$ in terms of quaternions and show how one can construct the vertices of 600 tetrahedra constituting the polytope {3, 3, 5}. We also prove why 20 tetrahedra meet at one point and show that the centres of these 20 tetrahedra constitute the vertices of a dodecahedron forming 120 cells of the polytope {5, 3, 3}. Section 4 is devoted to a discussion about the construction of some of the non-convex versions of these dual polytopes.

2. The Coxeter–Weyl group $W(SU(5))$ as a subgroup of $W(H_4)$ and 600 tetrahedra in {3, 3, 5}

Let $q = q_0 + q_i e_i$ $(i = 1, 2, 3)$ be a real unit quaternion with its conjugate defined by $\bar{q} = q_0 - q_i e_i$ and the norm $q\bar{q} = \bar{q}q = 1$. Here, the quaternionic imaginary units satisfy the relations

$$e_i e_i = -\delta_{ij} + \epsilon_{ijk} e_k \qquad (i, j, k = 1, 2, 3), \tag{1}$$

where δ_{ij} and ϵ_{ijk} are the Kronecker and Levi-Civita symbols, respectively, and summation over the repeated indices is implicit. With the definition of the scalar product

$$(p, q) = \tfrac{1}{2}(\bar{p}q + \bar{q}p), \tag{2}$$

quaternions generate the four-dimensional Euclidean space. The group of quaternions is isomorphic to $SU(2)$ which is a double cover of the proper rotation group $SO(3)$. Its finite subgroups are well studied [13, 14]. It has an infinite number of cyclic and dicyclic groups in addition to the binary tetrahedral group T, binary octahedral group O and binary icosahedral group I. In crystallography, these groups are called the double groups. An orthogonal rotation in 4D Euclidean space can be represented by the group elements of $O(4)$ as [5, 14]

$$[a, b] : q \to q' = aqb, \qquad [c, d]^{\star} : q \to q'' = c\bar{q}d. \tag{3}$$

When the quaternions p and q take values from the binary icosahedral group I, elements of which are classified in the form of conjugacy classes in table 1, the set of elements

$$W(H_4) = \{[p, q] \oplus [p, q]^{\star}\} \tag{4}$$

represents the Coxeter group $W(H_4)$ of order 14 400. In an earlier paper [12], we proved that the group $W(H_4)$ possesses five maximal subgroups, the group $[W(H_2) \times W(H_2)] : Z_4$ of order 400, the group $[W(SU(3)) \times W(SU(3))] : Z_4$ of order 144, the group $W(SO(8)) : Z_3$ of order 576, the group $W(H_3) \times Z_2$ and finally the group $\mathrm{Aut}(SU(5)) = W(SU(5)) : Z_2$ of orders 240. They seem to be crucial in the study of cell structures of the polytopes as well as for the construction of the anomalous polytopes. We will show that the duality properties of the

Figure 1. The Coxeter–Dynkin diagram of $SU(5)$ with quaternionic simple roots.

Table 1. Conjugacy classes of the binary icosahedral group I represented by quaternions.

Conjugacy classes and orders of elements	Elements of conjugacy classes also denoted by the numbers (cyclic permutations of e_1, e_2, e_3 must be added if not included)
1	1
2	-1
10	$12_+ : \frac{1}{2}(\tau \pm e_1 \pm \sigma e_3)$
5	$12_- : \frac{1}{2}(-\tau \pm e_1 \pm \sigma e_3)$
10	$12_+ : \frac{1}{2}(\sigma \pm e_1 \pm \tau e_2)$
5	$12'_- : \frac{1}{2}(-\sigma \pm e_1 \pm \tau e_2)$
6	$20_+ : \frac{1}{2}(1 \pm e_1 \pm e_2 \pm e_3), (1 \pm \tau e_1 \pm \sigma e_2)$
3	$20_- : \frac{1}{2}(-1 \pm e_1 \pm e_2 \pm e_3), (-1 \pm \tau e_1 \pm \sigma e_2)$
4	$30 : \pm e_1, \pm e_2, \pm e_3, \frac{1}{2}(\pm \sigma e_1 \pm \tau e_2 \pm e_3)$

polytopes $\{3, 3, 5\}$ and $\{5, 3, 3\}$ can be explained with the use of subgroups $W(SU(4)) \approx S_4$, a symmetry group of tetrahedron of order 24, acting on the vertices of $\{3, 3, 5\}$ and the icosahedral group $W(H_3)$ acting on the vertices of the polytope $\{5, 3, 3\}$. Before we study the action of $W(SU(4))$ on quaternions, we first study its five-fold embeddings in $W(SU(5)) \approx S_5$ which can be embeddable in $W(H_4)$ 120 ways. The Coxeter–Dynkin diagram of $SU(5)$ represented by quaternionic simple roots is shown in figure 1.

The generators $r_1 = [1, -1]^\star$, $r_2 = \left[\frac{1}{2}(1+e_1+e_2+e_3), -\frac{1}{2}(1+e_1+e_2+e_3)\right]^\star$, $r_3 = [e_1, -e_1]^\star$ and $r_4 = \left[\frac{1}{2}(e_1 - \sigma e_2 - \tau e_3), -\frac{1}{2}(e_1 - \sigma e_2 - \tau e_3)\right]^\star$ represent the reflection generators of $W(SU(5)) \approx S_5$ with respect to the hyperplanes orthogonal to the simple roots

$$\alpha_1 = -1, \qquad \alpha_2 = \frac{1}{2}(1 + e_1 + e_2 + e_3),$$
$$\alpha_3 = -e_1, \qquad \alpha_4 = \frac{1}{2}(e_1 - \sigma e_2 - \tau e_3). \tag{5}$$

Here $\tau = \frac{1+\sqrt{5}}{2}, \sigma = \frac{1-\sqrt{5}}{2}$ satisfy the relations $\tau\sigma = -1, \tau + \sigma = 1, \tau^2 = \tau + 1$ and $\sigma^2 = \sigma + 1$. The elements of the group $W(SU(5))$ can be compactly written in the form

$$W(SU(5)) = \{[p, \bar{c}\bar{p}^\dagger c] \oplus [p, c\bar{p}^\dagger c]^\star\}. \tag{6}$$

Here $p \in I$ is an arbitrary element of the binary icosahedral group I with $c = \frac{1}{\sqrt{2}}(e_2 - e_3)$, and $p^\dagger = p(\tau \leftrightarrow \sigma)$ is an element of the representation of the binary icosahedral group $I^\dagger$ obtained from I by interchanging τ and σ. The extended Coxeter–Dynkin diagram, playing a crucial role in this section, is depicted in figure 2. The element $c = \frac{1}{\sqrt{2}}(e_2 - e_3)$ belongs to the set of quaternions T' which consist of the elements

$$T' = \left\{\frac{1}{\sqrt{2}}(\pm 1 \pm e_1), \frac{1}{\sqrt{2}}(\pm e_2 \pm e_3), \frac{1}{\sqrt{2}}(\pm 1 \pm e_2),\right.$$
$$\left.\frac{1}{\sqrt{2}}(\pm e_3 \pm e_1), \frac{1}{\sqrt{2}}(\pm 1 \pm e_3), \frac{1}{\sqrt{2}}(\pm e_1 \pm e_2)\right\}. \tag{7}$$

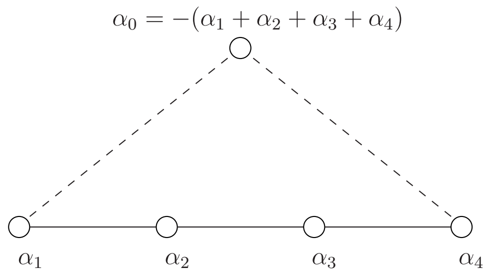

Figure 2. The extended Coxeter diagram of $SU(5)$ with quaternionic simple roots.

It represents the vertices of the polytope $\{3, 4, 3\}$ and together with its dual, the binary tetrahedral group represented by the elements

$$T = \left\{\pm 1, \pm e_1, \pm e_2, \pm e_3, \tfrac{1}{2}(\pm 1 \pm e_1 \pm e_2 \pm e_3)\right\}, \tag{8}$$

they form the binary octahedral group $O = T \oplus T'$ [15]. Let $t' \in T'$ be an arbitrary element. It can be shown that

$$\bar{t}' p t' = q^{\dagger}, \tag{9}$$

where $p \in I$ and $q^{\dagger} \in I^{\dagger}$.

The subgroup $W(SU(4))$ can be embedded in $W(SU(5))$ five different ways. The sets of generators of five conjugate subgroups $W(SU(4))$ can be classified as (r_1, r_2, r_3), (r_2, r_3, r_4), (r_3, r_4, r_0), (r_4, r_0, r_1) and (r_0, r_1, r_2), where $r_0 = [\alpha_0, -\alpha_0]^{\star}$ is the reflection generator with respect to the hyperplane represented by the root $\alpha_0 = -(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4) = \frac{1}{2}(1 - \tau e_2 - \sigma e_3)$. Let us denote $d = r_1 r_2 r_3 r_4 = [\alpha, \beta]$ by the Coxeter element of $W(SU(5))$ satisfying the relation $d^5 = [1, 1]$ with the elements $\alpha = \frac{1}{2}(-\sigma + e_1 + \tau e_2)$ and $\beta = \frac{1}{2}(-\tau + e_1 + \sigma e_2) = -c\bar{\alpha}^{\dagger} c$. The Coxeter element d permutes the five simple roots in figure 2, and the group generators $r_i (i = 0, 1, 2, 3, 4)$ are permuted by d-conjugation defined by $\mathrm{d} r_i d^{-1} = r_{i+1}$ (mod 4). This proves that the five conjugate groups of $W(SU(4))$ are permuted to each other under the d-conjugation. Therefore it is sufficient to study only one of them, say, the group generated by (r_1, r_2, r_3). Before we proceed further, it is imperative to discuss the embeddings of $W(SU(5))$ in $W(H_4)$. One possible representation of $W(SU(5))$ in $W(H_4)$ is already given in (6) which follows from the simple roots of figure 1. Since the index of $W(SU(5))$ in $W(H_4)$ is 120, it really means that one can represent the simple roots of $SU(5)$ 120 different ways. This can be achieved by multiplying the quaternionic roots of figure 1 from right or left with the elements of I because the quaternion multiplication preserves the scalar product (2). Let the new simple roots be represented by $\alpha_i q (i = 0, 1, 2, 3, 4)$, leading to the reflection generators

$$[\alpha_i q, -\alpha_i q]^{\star} = [1, q][\alpha_i, -\alpha_i]^{\star}[1, \bar{q}]. \tag{10}$$

Its action on the set of group elements (6) of $W(SU(5))$ will be

$$W'(SU(5)) = [1, q] W(SU(5)) [1, \bar{q}] = \{[p, (\bar{q}\bar{c})\bar{p}^{\dagger} cq] \oplus [p, cq^{\dagger}\bar{p}^{\dagger} cq]^{\star}\}. \tag{11}$$

Here, p and q are arbitrary elements of I. The Coxeter–Dynkin diagram symmetry allows us to write the maximal subgroup as

$$\mathrm{Aut}(SU(5)) = W(SU(5)) : Z_2 = \{[p, \pm(\bar{q}\bar{c})\bar{p}^{\dagger} cq] \oplus [p, \pm cq^{\dagger}\bar{p}^{\dagger} cq]^{\star}\}. \tag{12}$$

This proves that $\mathrm{Aut}(SU(5))$ can be embedded in 60 different ways, while $W(SU(5))$ can be represented in 120 ways in $W(H_4)$. It can be proven that a left multiplication leads to the same result in (12). Therefore, the arbitrary conjugate of $W(SU(5))$ in $W(H_4)$ is one of those in (11).

Construction of the vertices of $\{5, 3, 3\}$ *with the use of 5-cells*

The weights of the five-dimensional irreducible representation of $SU(5)$ can be computed with the usual Lie algebraic technique [16]. The highest weight (1000) of the irreducible representation $\underline{5}$ can be written in terms of the simple roots represented by the quaternions as

$$(1000) = \tfrac{1}{5}(4\alpha_1 + 3\alpha_2 + 2\alpha_3 + \alpha_4). \tag{13}$$

This can be normalized to the unit quaternion $\frac{1}{2\sqrt{2}}((\tau - \sigma) - \tau e_2 + \sigma e_3)$. Either applying the Coxeter–Weyl group $W(SU(5))$ to (13) or using the usual technique described in [16], one obtains the weights of $\underline{5}$ as

$$\begin{array}{lll} \frac{1}{2\sqrt{2}}((\tau - \sigma) - \tau e_2 + \sigma e_3), & \frac{1}{2\sqrt{2}}((-\tau + \sigma) - \tau e_2 + \sigma e_3), & \\ \frac{1}{2\sqrt{2}}((\tau - \sigma)e_1 - \sigma e_2 + \tau e_3), & \frac{1}{2\sqrt{2}}((-\tau + \sigma)e_1 - \sigma e_2 + \tau e_3), & \frac{1}{\sqrt{2}}(e_2 - e_3). \end{array} \tag{14}$$

Recalling the definition of $c = \frac{1}{\sqrt{2}}(e_2 - e_3)$, we can represent the weights of $\underline{5}$ collectively as

$$\alpha^a c \beta^a = \alpha^a \bar{\alpha}^{\dagger a} c, \qquad (a = 0, 1, 2, 3, 4). \tag{15}$$

It can be proven that the quaternionic weights in (15) represent a 5-cell consisting of five tetrahedra. When we represent the simple roots of $SU(5)$ by $q\alpha_i$, $(i = 1, 2, 3, 4)$ for $q \in I$, then the weights of $\underline{5}$ can be put in the form $q\alpha^a \bar{\alpha}^{\dagger a} c$. It is interesting to note that the 600 vertices of the polytope $\{5, 3, 3\} = \sum_{a,b=0}^{4} \oplus\, \alpha^a T' \beta^b$ [10, 17] can be written in the form $\{5, 3, 3\} = \sum_{a=0}^{4} \oplus\, I\alpha^a \bar{\alpha}^{\dagger a} c$. This is an extremely useful result that allows us to interpret the 600 vertices of the polytope $\{5, 3, 3\}$, consisting of disjoint sets of vertices obtained by multiplying the quaternionic vertices of 5-cells with the elements of the binary icosahedral group. If we had multiplied the simple roots of $SU(5)$ on the right, it can be proven that [17] we obtain the same set of vertices. This is very much analogous to the case that a cube consists of two disjoint tetrahedra where the tetrahedra are transformed to each other by a Z_2 symmetry which is a normal subgroup of the symmetry group of the cube of order 48. Here, we have a similar structure where the 5-cells are transformed to each other by the normal subgroup $[I, 1]$ or $[1, I]$ of $W(H_4)$, the symmetry group of the polytope $\{5, 3, 3\}$.

600 tetrahedral cells of the polytope $\{3, 3, 5\}$

The vertices of the polytope $\{3, 3, 5\}$ are the quaternionic elements of the binary icosahedral group I tabulated in table 1. We will prove that it consists of 600 tetrahedra, 20 of which are sharing the same common point. The polytope $\{3, 3, 5\}$ is obtained by joining the nearest points. The shortest distance between two unit quaternions q_1 and q_2 taking values in I can be determined by demanding that $d = \sqrt{(q_1 - q_2)(\bar{q}_1 - \bar{q}_2)} = \sqrt{2 - (q_1\bar{q}_2 + q_2\bar{q}_1)}$ is the least among others. This means that $q_1\bar{q}_2 + q_2\bar{q}_1 = 2\cos\theta_{12}$ is the largest positive number among all possibilities, where $\cos\theta_{12}$ takes the values $\pm\frac{\tau}{2}, \pm\frac{\sigma}{2}, \pm\frac{1}{2}, 0$. Therefore, the choice is $\cos\theta_{12} = \frac{\tau}{2}$ leading to the shortest distance $d = -\sigma = \frac{\sqrt{5}-1}{2} \approx 0.618$ to obtain the convex polytope $\{3, 3, 5\}$. One can readily see that the scalar products of the quaternion 1 with the quaternions in the conjugacy class denoted by 12_+ satisfy this requirement. This also shows that each quaternion is joined to the nearest 12 quaternions within the polytope $\{3, 3, 5\}$, implying that the polytope has 720 edges. It can be shown that the quaternions in the conjugacy class 12_+ can be classified as 20 equilateral triangles with their vertices being

Table 2. The sets of tetrahedra whose quaternionic vertices obtained from the conjugacy classes 1 and 12_+.

1	$\frac{1}{2}(\tau+e_1+\sigma e_3)$	$\frac{1}{2}(\tau-e_3+\sigma e_2)$	$\frac{1}{2}(\tau-e_2-\sigma e_1)$
1	$\frac{1}{2}(\tau+e_1+\sigma e_3)$	$\frac{1}{2}(\tau-e_3-\sigma e_2)$	$\frac{1}{2}(\tau+e_2-\sigma e_1)$
1	$\frac{1}{2}(\tau+e_1+\sigma e_3)$	$\frac{1}{2}(\tau+e_2-\sigma e_1)$	$\frac{1}{2}(\tau+e_1-\sigma e_3)$
1	$\frac{1}{2}(\tau+e_1+\sigma e_3)$	$\frac{1}{2}(\tau-e_3-\sigma e_2)$	$\frac{1}{2}(\tau-e_3+\sigma e_2)$
1	$\frac{1}{2}(\tau+e_1+\sigma e_3)$	$\frac{1}{2}(\tau+e_1-\sigma e_3)$	$\frac{1}{2}(\tau-e_2-\sigma e_1)$
1	$\frac{1}{2}(\tau+e_2+\sigma e_1)$	$\frac{1}{2}(\tau-e_1-\sigma e_3)$	$\frac{1}{2}(\tau+e_3-\sigma e_2)$
1	$\frac{1}{2}(\tau+e_2+\sigma e_1)$	$\frac{1}{2}(\tau-e_1+\sigma e_3)$	$\frac{1}{2}(\tau-e_3-\sigma e_2)$
1	$\frac{1}{2}(\tau+e_2+\sigma e_1)$	$\frac{1}{2}(\tau-e_1-\sigma e_3)$	$\frac{1}{2}(\tau-e_1+\sigma e_3)$
1	$\frac{1}{2}(\tau+e_2+\sigma e_1)$	$\frac{1}{2}(\tau+e_2-\sigma e_1)$	$\frac{1}{2}(\tau-e_3-\sigma e_2)$
1	$\frac{1}{2}(\tau+e_2+\sigma e_1)$	$\frac{1}{2}(\tau+e_3-\sigma e_2)$	$\frac{1}{2}(\tau+e_2-\sigma e_1)$
1	$\frac{1}{2}(\tau+e_3+\sigma e_2)$	$\frac{1}{2}(\tau-e_2-\sigma e_1)$	$\frac{1}{2}(\tau+e_1-\sigma e_3)$
1	$\frac{1}{2}(\tau+e_3+\sigma e_2)$	$\frac{1}{2}(\tau-e_2+\sigma e_1)$	$\frac{1}{2}(\tau-e_1-\sigma e_3)$
1	$\frac{1}{2}(\tau+e_3+\sigma e_2)$	$\frac{1}{2}(\tau-e_2-\sigma e_1)$	$\frac{1}{2}(\tau-e_2+\sigma e_1)$
1	$\frac{1}{2}(\tau+e_3+\sigma e_2)$	$\frac{1}{2}(\tau+e_3-\sigma e_1)$	$\frac{1}{2}(\tau-e_1-\sigma e_3)$
1	$\frac{1}{2}(\tau+e_3+\sigma e_2)$	$\frac{1}{2}(\tau+e_1-\sigma e_3)$	$\frac{1}{2}(\tau+e_3-\sigma e_2)$
1	$\frac{1}{2}(\tau+e_1-\sigma e_3)$	$\frac{1}{2}(\tau+e_2-\sigma e_1)$	$\frac{1}{2}(\tau+e_3-\sigma e_2)$
1	$\frac{1}{2}(\tau-e_1+\sigma e_3)$	$\frac{1}{2}(\tau-e_2+\sigma e_1)$	$\frac{1}{2}(\tau-e_3+\sigma e_2)$
1	$\frac{1}{2}(\tau-e_1+\sigma e_3)$	$\frac{1}{2}(\tau-e_1-\sigma e_3)$	$\frac{1}{2}(\tau-e_2+\sigma e_1)$
1	$\frac{1}{2}(\tau-e_2+\sigma e_1)$	$\frac{1}{2}(\tau-e_2-\sigma e_1)$	$\frac{1}{2}(\tau-e_3+\sigma e_2)$
1	$\frac{1}{2}(\tau-e_1+\sigma e_3)$	$\frac{1}{2}(\tau-e_3+\sigma e_2)$	$\frac{1}{2}(\tau-e_3-\sigma e_2)$

equidistant from the quaternion 1. To give an example, we take the quaternions

$$1, \qquad \tfrac{1}{2}(\tau+e_1+\sigma e_3), \qquad \tfrac{1}{2}(\tau-e_3+\sigma e_2), \qquad \tfrac{1}{2}(\tau-e_2-\sigma e_1), \tag{16}$$

which constitute the vertices of a tetrahedron where the edge length is $-\sigma$. The easiest way to decompose the quaternions in the conjugacy class 12_+ into 20 equilateral triangles is to determine them as the orbits of the cyclic groups $Z_3 = [q, \bar{q}]$ with $q \in 20_+$ or $q \in 20_-$. The set of tetrahedra obtained in this way are listed in table 2.

The centres of these 20 tetrahedra constitute the vertices of a dodecahedron invariant under the icosahedral group $W(H_3) = \{[p, \bar{p}] \oplus [p, \bar{p}]^\star\}, p \in I$. This is an obvious fact because 1 and 12_+ are invariants under the icosahedral group. When the centres of these tetrahedra are normalized to unit quaternions, they read

$$\Big\{\tfrac{1}{2\sqrt{2}}(\tau^2 \pm e_1 \pm \sigma^2 e_2), \tfrac{1}{2\sqrt{2}}(\tau^2 \pm e_2 \pm \sigma^2 e_3), \\ \times \tfrac{1}{2\sqrt{2}}(\tau^2 \pm e_3 \pm \sigma^2 e_1), \tfrac{1}{2\sqrt{2}}(\tau^2 \pm \sigma e_1 \pm \sigma e_2 \pm \sigma e_3)\Big\}. \tag{17}$$

These are the vertices of a dodecahedron obtained by projecting the polytope {5, 3, 3} to three-dimensions as the orbit of the group $W(H_3)$ [10].

3. $W(SU(4))$ analysis of the polytope {3, 3, 5}

Let us consider the subgroup $W(SU(4))$ of $W(SU(5))$ generated by (r_1, r_2, r_3). The general group elements can be compactly written as

$$W(SU(4)) = \{[p, \bar{c}\bar{p}c] \oplus [p, c\bar{p}c]^\star\}, \tag{18}$$

where $p \in T$ is an arbitrary element of the binary tetrahedral group of (8), a subgroup of the binary icosahedral group I and $c = \frac{1}{\sqrt{2}}(e_2 - e_3)$. One can prove that the group represented by (18) is the discrete transformation isomorphic to the permutation group S_4 of order 24, a symmetry group of the tetrahedron, acting in the hyperplane orthogonal to the vector $c = \frac{1}{\sqrt{2}}(e_2 - e_3)$. A simple proof can be given by introducing the new generators $a = r_2 r_1$ and $b = r_1 r_2 r_3$ satisfying the generation relations $b^4 = a^3 = (ba)^2 = [1, 1]$, which was proven by Coxeter that the above relation is a presentation of the group S_4 [18]. Now we discuss how one can construct the vertices of a tetrahedron in terms of quaternions. One may start with the vector left invariant under the generator $a = r_2 r_1$. Let us apply it to an arbitrary unit quaternion $q = q_0 + q_i e_i$. When q is left invariant under the operation of $a = r_2 r_1$, it leads to the relations

$$q_0 = 0, \qquad q_3 = -(q_1 + q_2), \qquad q_1^2 + q_2^2 + q_1 q_2 = \tfrac{1}{2}. \tag{19}$$

Solutions of the equations in (19) restricted by the set of quaternions $I = \{3, 3, 5\}$ lead to the set of quaternions

$$q = \tfrac{1}{2}(\sigma e_1 + \tau e_2 - e_3), \qquad t = \tfrac{1}{2}(\tau e_1 + \sigma e_2 - e_3) \qquad \omega = \tfrac{1}{2}(-e_1 + \sigma e_2 + \tau e_3), \tag{20}$$

up to a change of the overall sign. Note that they are obtained from each other by the cyclic permutation of e_1, e_2 and e_3. Applying the generator $b = r_1 r_2 r_3$ to any one of the above quaternion, one generates four quaternions which will represent the vertices of a tetrahedron. For instance, when $b = r_1 r_2 r_3$ is applied on the first quaternion of (20), one generates the quaternions

$$\begin{aligned} q_1 &= \tfrac{1}{2}(-\sigma + e_2 - \tau e_3), \qquad & q_2 &= \tfrac{1}{2}(\sigma + e_2 - \tau e_3), \\ q_3 &= \tfrac{1}{2}(-\sigma e_1 + \tau e_2 - e_3), \qquad & q_4 &= \tfrac{1}{2}(\sigma e_1 + \tau e_2 - e_3), \end{aligned} \tag{21}$$

forming the vertices of a tetrahedron. Those quaternions with the negative signs of the above quaternions form another tetrahedron in four dimensions. One can generate the vertices of two more sets of quaternions (with the negative signs, one has four tetrahedra) when the generator $b = r_1 r_2 r_3$ is applied to the other quaternions in (20). However, we will restrict to the tetrahedron represented by the vertices in (21) because they satisfy the shortest distance criteria $\cos\theta_{12} = \frac{\tau}{2}$ required by a convex polytope. The other two sets of tetrahedra will lead to the non-convex versions of the polytopes {3, 3, 5} and {5, 3, 3}, which will not be discussed in this paper. We have already noted that $W(SU(4))$ can be embedded in $W(SU(5))$ five different ways. One can prove that these five different representations of $W(SU(4))$ in $W(SU(5))$ can be obtained by computing the conjugate groups as follows:

$$[\alpha, \beta] W(SU(4)) [\alpha, \beta]^{-1}. \tag{22}$$

This leads to five conjugate groups represented by

$$W(SU(4))^{(a)} = \{[T^a, \bar{c}^a \bar{T}^a c^a] \oplus [T^a, c^a \bar{T}^a c^a]^\star\} \qquad (a = 0, 1, 2, 3, 4). \tag{23}$$

Here we have used the notations $T^a = \alpha^a T \bar{\alpha}^a$, $c^a = \alpha^a c \beta^a = \alpha^a \bar{\alpha}^{\dagger a} c$, where T stands for an arbitrary element of the binary tetrahedral group. Note that $c^a = \alpha^a \bar{\alpha}^{\dagger a} c$ represent the weights of the irreducible representation $\underline{5}$ of $SU(5)$.

When we calculate the highest weight (100) of the four-dimensional irreducible representation of $SU(4)$, we obtain

$$(100) = \tfrac{1}{4}(3\alpha_1 + 2\alpha_2 + \alpha_3) = \tfrac{1}{4}(-2 + e_2 + e_3). \tag{24}$$

Either applying the group elements in (18) or using the usual Lie algebraic methods, one obtains the weights of the irreducible representation $\underline{4}$ of $SU(4)$ as follows:

$$\begin{aligned} &\tfrac{1}{4}(-2 + e_2 + e_3), \qquad && \tfrac{1}{4}(2 + e_2 + e_3), \\ &\tfrac{1}{4}(-2e_1 - e_2 - e_3), \qquad && \tfrac{1}{4}(2e_1 - e_2 - e_3). \end{aligned} \tag{25}$$

They form the vertices of a tetrahedron whose centre is at the origin. The question is now how these weights are related to the vertices in (21). The quaternions in (25) can be related to the quaternions in (21) by adding the invariant vector $(e_2 - e_3)$ in the following way:

$$\frac{x}{4}(-2+e_2+e_3)+y(e_2-e_3)=q_1 \qquad \frac{x}{4}(2+e_2+e_3)+y(e_2-e_3)=q_2$$
$$\frac{x}{4}(2e_1-e_2-e_3)+y(e_2-e_3)=q_3 \qquad \frac{x}{4}(2e_1-e_2-e_3)+y(e_2-e_3)=q_4. \tag{26}$$

We have a unique solution of the above equation with $x = \sigma$ and $y = \frac{\tau^2}{4}$. Of course, adding a constant vector to those in (25) shifts the centre of the tetrahedron to the point $\frac{\tau^2}{4}(e_2 - e_3)$. There are two more tetrahedra with the centres proportional to $(e_2 - e_3)$, which will lead to non-convex polytopes. Although we will not be working with the non-convex polytopes in this paper, we will list those quaternions representing the vertices of the other two tetrahedra. Similar to the equations of (26), we can write

$$\begin{aligned}
&\frac{x}{4}(-2+e_2+e_3)+y(e_2-e_3)=t_1=\frac{1}{2}(-\tau-\sigma e_2+e_3)\\
&\frac{x}{4}(2+e_2+e_3)+y(e_2-e_3)=t_2=\frac{1}{2}(\tau-\sigma e_2+e_3)\\
&\frac{x}{4}(-2e_1-e_2-e_3)+y(e_2-e_3)=t_3=\frac{1}{2}(-\tau e_1-e_2+\sigma e_3)\\
&\frac{x}{4}(2e_1-e_2-e_3)+y(e_2-e_3)=t_4=\frac{1}{2}(\tau e_1-e_2+\sigma e_3)
\end{aligned} \tag{27}$$

with the solutions $x = \tau$, $y = -\frac{\sigma^2}{4}$ and

$$\begin{aligned}
&\frac{x}{4}(-2+e_2+e_3)+y(e_2-e_3)=\omega_1=\frac{1}{2}(1-\tau e_2-\sigma e_3)\\
&\frac{x}{4}(2+e_2+e_3)+y(e_2-e_3)=\omega_2=\frac{1}{2}(1+\tau e_2+\sigma e_3)\\
&\frac{x}{4}(2e_1-e_2-e_3)+y(e_2-e_3)=\omega_3=\frac{1}{2}(e_1+\sigma e_2+\tau e_3)\\
&\frac{x}{4}(2e_1-e_2-e_3)+y(e_2-e_3)=\omega_4=\frac{1}{2}(-e_1+\sigma e_2+\tau e_3)
\end{aligned} \tag{28}$$

with the solutions $x = -1$ and $y = \frac{-\tau+\sigma}{4}$. We note that the quaternions in (27) and (28) satisfy the relations $\cos\theta_{ij} = \frac{\sigma}{2}$ and $\cos\theta_{ij} = \frac{1}{2}$, respectively.

Now we apply the Coxeter element $d = [\alpha, \beta]$ of $W(SU(5))$ on the vectors in (21) to generate five tetrahedra

$$\alpha^a q_i \beta^a \ (i = 1, 2, 3, 4), \qquad (a = 0, 1, 2, 3, 4). \tag{29}$$

Note that for each value of a, we have one tetrahedron. Using the quaternions of (29), we can construct those vertices of 20 tetrahedra listed in table 2 as follows. Let us start with the quaternions with $a = 0$, namely, with the set $q_i (i = 1, 2, 3, 4)$ given in (21). One can construct the set of vectors

$$\bar{q}_1 q_1 = 1, \qquad \bar{q}_1 q_2 = \tfrac{1}{2}(\tau - \sigma e_2 - e_3),$$
$$\bar{q}_1 q_3 = \tfrac{1}{2}(\tau + e_2 + \sigma e_1), \qquad \bar{q}_1 q_4 = \tfrac{1}{2}(\tau - e_1 - \sigma e_3). \tag{30}$$

This set of vectors represents the vertices of a tetrahedron just like those in (21) do since the multiplication of the quaternions in (21) by a fixed quaternion will not alter the scalar product between the vectors in (21). The set of quaternions in (30) is just one set of those vertices of 20 tetrahedra in table 2. To construct the others, we proceed as follows. One can multiply

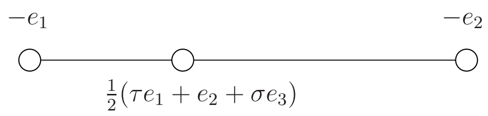

Figure 3. The Coxeter diagram of H_3 with quaternionic simple roots.

the set of vectors in (21) with any $\bar{q}_i$ $(i = 1, 2, 3, 4)$ to obtain the set $\sum_{i,j=1}^{4} \oplus \bar{q}_i q_j$, which represents the four tetrahedra in table 2. More generally, we can construct the set of vectors

$$\sum_{a=0}^{4} \left(\sum_{i,j=1}^{4} \oplus \bar{\beta}^a \bar{q}_i \bar{\alpha}^a \alpha^a q_j \beta^a \right) = \sum_{a=0}^{4} \left(\sum_{i,j=1}^{4} \oplus \bar{\beta}^a \bar{q}_i q_j \beta^a \right), \tag{31}$$

which represent those 20 quaternions in table 2. Centres of these tetrahedra can be obtained with an average taken over the index j, which leads to the vertices of the dodecahedron in (17) up to some normalization. Now, we look at the problem from two different points of views. The quaternions in (29) and (31) are obtained from a $SU(5)$ diagram. We have already discussed that $W(SU(5))$ can be embedded in $W(H_4)$ 120 different ways by multiplying the simple roots of $SU(5)$ by the elements of the binary icosahedral group I. Therefore, those quaternions in (29) representing the vertices of five tetrahedra can be constructed in 120 different ways leading to the replacement of q_i's with 120 q_i's and $d = [\alpha, \beta]$ with its 120 conjugates. Since the quaternions in (29) are the elements of I and those to be constructed will also belong to the set I, then the polytope $I = \{3, 3, 5\}$ consists of 600 tetrahedra as we have discussed in section 2. The second view is that the 20 quaternions in (31) leading to the vertices of a dodecahedron when averaged over the index j is invariant under the icosahedral group $W(H_3)$ represented by

$$W(H_3) = \{[p, \bar{p}] \oplus [p, \bar{p}]^\star\} \approx A_5 \times Z_2, \qquad p \in I. \tag{32}$$

Here $A_5 = [p, \bar{p}]$, $p \in I$ is the icosahedral group without inversion and $Z_2 = [1, 1]^\star$ is representing the inversion in three dimensions. This representation of $W(H_3)$ can be derived from the Coxeter diagram of H_3 given in figure 3. Note that when the average over the index j is carried out, the result would read $\sum_{a=0}^{4} \sum_{i=1}^{4} \oplus \frac{\tau^2}{4} \bar{\beta}^a \bar{q}_i c \beta^a$. Here we have five tetrahedra embedded in a dodecahedron because $\bar{\beta}^a \bar{q}_i c \beta^a$ represent the five tetrahedra, one for each a. Multiplying each set by I on the right leads to $120 \times 20 = 2400$ quaternions; however, it can be proven that the number of distinct vertices are 600. This also indicates that four dodecahedra meet at one point.

Since the simple roots are chosen from I we can construct the Coxeter diagram H_3 as many different ways as the index of $W(H_3)$ in $W(H_4)$, namely in 120 ways. This can be achieved by multiplying the simple roots of H_3 from left or equivalently from right by the elements $q \in I$. Then the generators of $W(H_3)'$

$$r_i' = [\alpha_i q, -\alpha_i q]^\star \tag{33}$$

would lead to the representation

$$W(H_3)' = \{[p, \bar{q}\bar{p}q] \oplus [p, q\bar{p}q]^\star\}; \qquad p, q \in I. \tag{34}$$

This representation of $W(H_3)$ leaves the vector $\pm q$ invariant, where (32) is the special one leaving ± 1 invariant. The maximal subgroup $W(H_3) \times Z_2$ which can be embedded in 60 different ways in $W(H_4)$ will be represented by

$$W(H_3) \times Z_2 = \{[p, \pm\bar{q}\bar{p}q] \oplus [p, \pm q\bar{p}q]^\star\}; \qquad p, q \in I. \tag{35}$$

Actually the group $W(H_3)$ not only leaves the dodecahedron in (17) invariant, but also the set of vectors which are the negatives of (17). Therefore for a given vector $\pm q$ we have two sets of dodecahedra implying that we can construct the 120 dodecahedra of the polytope {5, 3, 3} by multiplying the quaternions in (17) with any element of I on the right or on the left. It is certainly true that the centres of the 120 dodecahedra constitute the vertices of the polytope $I = \{3, 3, 5\}$ up to a normalization.

4. Concluding remarks

In this paper, we have explicitly studied the cell structures of the convex polytopes {3, 3, 5} and its dual {5, 3, 3} using the quaternions. The relevant symmetries as well as the vertices of the polytopes are constructed in terms of quaternions. Of course, we have not worked out the non-convex versions of these polytopes. This topic could be a subject of research by itself within the context of quaternionic representations. However, here, we wish to comment on the fact that we have obtained three types of tetrahedra with edge lengths 0.618, 1 and 1.618 when we applied the elements of $W(SU(4))$ on the elements of the binary icosahedral group I. In[10] when we projected the polytope {5, 3, 3} in three dimensions by slicing the polytope with parallel hyperplanes, we had obtained three dodecahedra (actually six dodecahedra for $Sc(q) = q_0 = \pm\frac{\tau^2}{2\sqrt{2}}, \pm\frac{\sigma^2}{2\sqrt{2}}, \pm\frac{\tau-\sigma}{2\sqrt{2}}$). That was a surprise for us because the projection was made on the hyperplanes orthogonal to the quaternion ± 1. One normally expects one dodecahedron for each sign, whereas we had three dodecahedra for each sign. Now it is clear that they arise from the fact that the set of vectors of the binary icosahedral group I represent not only the convex polytope $I = \{3, 3, 5\}$, but also two more polytopes made of 600 tetrahedra with longer edge lengths 1 and 1.618. Existence of two more tetrahedra with different edge lengths can be interpreted that the set of vectors of I constitutes non-convex versions of the 600-cell whose duals will correspond to the non-convex versions of the 120-cell.

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